

Lie super-bialgebra structures on a class of generalized super W -algebra \mathfrak{L}

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Abstract. In this paper, Lie super-bialgebra structures on a class of generalized super W -algebra \mathfrak{L} are investigated. By proving the first cohomology group of \mathfrak{L} with coefficients in its adjoint tensor module is trivial, namely, $H^1(\mathfrak{L}, \mathfrak{L} \otimes \mathfrak{L}) = 0$, we obtain that all Lie super-bialgebra structures on \mathfrak{L} are triangular coboundary.

Key words: cohomology group, generalized super W -algebras, Lie super-bialgebra

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1 Introduction

It is well known that the Virasoro algebra (named after the physicist Miguel Angel Virasoro) is a very important infinite dimensional Lie algebra and is widely used in conformal field theory and string theory. After that much attention has been paid to the Virasoro type Lie algebras or superalgebras (which contains the Virasoro algebra as its subalgebra), including their construction, structures and representations. The W -algebra $W(2, 2)$ is certainly a Virasoro type Lie algebra, which plays important rolls in many areas of mathematics and physics (which was introduced in [24] during the study of vertex operator algebras). It possesses a basis $\{L_m, I_m | m \in \mathbb{Z}\}$ as a vector space over the complex field \mathbb{C} , with the Lie brackets $[L_m, L_n] = (m - n)L_{m+n}$, $[L_m, I_n] = (m - n)I_{m+n}$, $[I_m, I_n] = 0$. Structures and representations of $W(2, 2)$ are extensively investigated in many references, such as [1], [6], [8], [9], [10] and [25].

Some Lie superalgebras with W -algebra $W(2, 2)$ as their even parts were constructed in [20] as an application of the classification of Balinsky-Novikov super-algebras with dimension $2|2$. The *generalized super W -algebra* $W(2, 2)$ whose even part is the generalized W -algebra $W(2, 2)$ is an infinite-dimensional Lie super algebra with the \mathbb{C} -basis $\{L_p, I_p, G_r, H_r | p \in \Gamma, r \in s + \Gamma\}$, where Γ is a nontrivial abelian subgroup of \mathbb{R} , $2s \in \Gamma$, admitting the following non-vanishing super-brackets:

$$\begin{aligned} [L_p, L_q] &= (p - q)L_{p+q}, & [L_p, I_q] &= (p - q)I_{p+q}, \\ [L_p, H_r] &= (\frac{p}{2} - r)H_{p+r}, & [G_r, G_t] &= I_{r+t}, \\ [L_p, G_r] &= (\frac{p}{2} - r)G_{p+r}, & [I_p, G_r] &= (p - 2r)H_{p+r}. \end{aligned} \tag{1.1}$$

For convenience, we denote such an algebra by \mathfrak{L} .

In this paper, we investigated the Lie super-bialgebra structures of \mathfrak{L} , and proved that all Lie super bialgebra structures on \mathfrak{L} are triangular coboundary (see Theorem 2.3). Our motivations mainly originate from the following.

- To construct Lie super bialgebras and their quantizations is an important approach to produce new quantum groups. Since the notion of Lie bialgebras was introduced by Drinfeld in 1983 (Refs. [2, 3]), there have appeared several papers on Lie coalgebras or Lie (super) bialgebras (e.g., Refs. [4, 5, 9, 11–17, 19, 21–23]).
- Though the result that all Lie super bialgebra structures on \mathfrak{L} are triangular coboundary is not surprising, and coboundary triangular Lie super bialgebras have relatively simple structures, it seems to us that it is still worth paying more attention on them, as one can see from [18] that by considering dual structures of Lie bialgebras, one may expect to obtain some new Lie algebras, which is our next goal.

Finally we would like to make some remarks. We observe that many papers are forced on the Virasoro type Lie superalgebras which contain the super Virasoro algebra as their subalgebra (e.g., Refs. [4, 23]), especially the $N = 2$ super Virasoro algebras. It is easy to find that the algebra \mathfrak{L} doesn't contain the super Virasoro Lie algebra as its subalgebra, so the methods developed there are not applicable to \mathfrak{L} . What's more, the subscript set Γ in our algebra is an arbitrary nontrivial abelian subgroup of \mathbb{R} , not necessarily discrete. In other words, it is possible that we can't find a minimal positive element in Γ . All these make the study of \mathfrak{L} more challengeable and attractive (this is also one of our motivations to present our results here), we need to find some new methods to handel these problems. For instance, one of our strategies used in the present paper is to introduce the length of a derivation so that the determination of derivations can be done by induction on the length. We would also like to mention that although central extension makes the representations of \mathfrak{L} more interesting, it makes only little difference about the bialgebra structures. So we only consider the centerless case here. The investigation of central extension of \mathfrak{L} and its representation theory is also our next goal.

2 The main results

We briefly recall some notations of Lie super-bialgebras. Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a vector space over \mathbb{C} , and all elements below are assumed to be \mathbb{Z}_2 -homogeneous in this subsection, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For any homogeneous element $x \in L$, we always denote by $[x] \in \mathbb{Z}_2$ the *parity* of x , i.e., $x \in L_{[x]}$. Throughout what follows, if $[x]$ occurs in an expression, then it is assumed that x is homogeneous and that the expression extends to the other elements by linearity. Denote by τ the *super-twist map* of $L \otimes L$:

$$\tau(x \otimes y) = (-1)^{[x][y]} y \otimes x, \quad \forall x, y \in L.$$

Denote by ξ the *super-cyclic map* which cyclically permutes the coordinates in $L \otimes L \otimes L$:

$$\xi = (1 \otimes \tau) \cdot (\tau \otimes 1) : x_1 \otimes x_2 \otimes x_3 \mapsto (-1)^{[x_1]([x_2]+[x_3])} x_2 \otimes x_3 \otimes x_1,$$

for all $x_1, x_2, x_3 \in L$, where 1 is the identity map of L . Then we can rewrite the definition of Lie super-algebra as follows: A *Lie super-algebra* is a pair (L, φ) consisting of super-vector space L and a bilinear map $\varphi : L \otimes L \rightarrow L$ (the *super-bracket*) satisfying the following conditions:

$$\begin{aligned} \varphi(L_i, L_j) &\subset L_{i+j}, \quad \text{Ker}(1 \otimes 1 - \tau) \subset \text{Ker}\varphi, \\ \varphi \cdot (1 \otimes \varphi) \cdot (1 \otimes 1 \otimes 1 + \xi + \xi^2) &= 0 : L \otimes L \otimes L \rightarrow L. \end{aligned}$$

Definition 2.1. (1) A *Lie super-coalgebra* is a pair (L, Δ) consisting of a super-vector space L and a linear map $\Delta : L \rightarrow L \otimes L$ (the *super-cobacket*) satisfying

$$\begin{aligned} \Delta(L_i) &\subset \sum_{j+k=i} L_j \otimes L_k, \quad \text{Im}\Delta \subset \text{Im}(1 \otimes 1 - \tau), \\ (1 \otimes 1 \otimes 1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta &= 0 : L \longrightarrow L \otimes L \otimes L. \end{aligned}$$

(2) A *Lie super-bialgebra* is a triple (L, φ, Δ) satisfying

(i) (L, φ) is a *Lie super-algebra*,

(ii) (L, Δ) is a *Lie super-coalgebra*,

(iii) $\Delta\varphi(x \otimes y) = x \circ \Delta(y) - (-1)^{[x][y]} y \circ \Delta(x)$ for all $x, y \in L$, where the symbol \circ means the adjoint diagonal action:

$$x \circ (\sum_i (a_i \otimes b_i)) = \sum_i ([x, a_i] \otimes b_i + (-1)^{[x][a_i]} a_i \otimes [x, b_i]), \quad \forall x, a_i, b_i \in L. \quad (2.1)$$

Definition 2.2. (1) A *coboundary Lie super-bialgebra* is a quadruple (L, φ, Δ, r) where (L, φ, Δ) is a *Lie super-bialgebra* and $r \in \text{Im}(1 \otimes 1 - \tau) \subset L \otimes L$ such that $\Delta = \Delta_r$ is a coboundary of r , where in general Δ_r is defined by

$$\Delta_r(x) = (-1)^{[r][x]} x \circ r, \quad \forall x \in L. \quad (2.2)$$

(2) A *coboundary Lie super-bialgebra* (L, φ, Δ, r) is called *triangular* if it satisfies the classical Yang-Baxter equation (CYBE):

$$c(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \quad (2.3)$$

where r^{ij} are defined by (2.5).

An element r in a Lie super-algebra L is said to satisfy the modified Yang-Baxter equation (MYBE) if

$$x \circ c(r) = 0, \quad \forall x \in L. \quad (2.4)$$

Denote by $\mathcal{U}(L)$ the universal enveloping algebra of L . If $r = \sum_i a_i \otimes b_i \in L \otimes L$, then (here we also use 1 to denote the unit element in $\mathcal{U}(L)$):

$$\begin{aligned} r^{12} &= \sum_i a_i \otimes b_i \otimes 1 = r \otimes 1, \\ r^{13} &= \sum_i a_i \otimes 1 \otimes b_i = (\tau \otimes 1)(1 \otimes r), \\ r^{23} &= \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes r \end{aligned} \tag{2.5}$$

are elements of $\mathcal{U}(L) \otimes \mathcal{U}(L) \otimes \mathcal{U}(L)$. Obviously,

$$\begin{aligned} [r^{12}, r^{13}] &= \sum_{i,j} (-1)^{[a_j][b_i]} [a_i, a_j] \otimes b_i \otimes b_j, \\ [r^{12}, r^{23}] &= \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j, \\ [r^{13}, r^{23}] &= \sum_{i,j} (-1)^{[a_j][b_i]} a_i \otimes a_j \otimes [b_i, b_j] \end{aligned} \tag{2.6}$$

are elements of $L \otimes L \otimes L$.

For a Lie super-algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be an L -module. An \mathbb{Z}_2 -homogeneous linear map $d : L \rightarrow V$ such that there exists $[d] \in \mathbb{Z}_2$ with $d(L_i) \in V_{i+[d]}$ for $i \in \mathbb{Z}_2$ and

$$d([x, y]) = (-1)^{[d][x]} x \circ d(y) - (-1)^{[y]([d]+[x])} y \circ d(x), \quad \forall x, y \in L, \tag{2.7}$$

is called a homogeneous derivation of parity $[d] \in \mathbb{Z}_2$. The derivation d is called even if $[d] = \bar{0}$ and odd if $[d] = \bar{1}$. Denote by $\text{Der}_p(L, V)$ the set of homogeneous derivations of parity p ($p \in \{\bar{0}, \bar{1}\}$) and $\text{Der}(L, V) = \text{Der}_{\bar{0}}(L, V) \oplus \text{Der}_{\bar{1}}(L, V)$ the set of derivations from L to V . Denote by $\text{Inn}(L, V) = \text{Inn}_{\bar{0}}(L, V) \oplus \text{Inn}_{\bar{1}}(L, V)$ the set of all inner derivations from L to V , where $\text{Inn}_p(L, V)$ is the set of homogeneous inner derivations of parity p consisting of a_{Inn} ($a \in V_p$) defined by

$$a_{\text{Inn}}(x) = (-1)^{[a][x]} x \circ a, \quad \forall x \in L, [a] = p. \tag{2.8}$$

Denote by $H^1(L, V)$ the first cohomology group of L with coefficients in V , then it is known

$$H^1(L, V) \cong \text{Der}(L, V) / \text{Inn}(L, V). \tag{2.9}$$

The main results of this article can be formulated as the following theorem.

Theorem 2.3. (1) $H^1(\mathfrak{L}, \mathfrak{V}) = 0$, where $\mathfrak{V} = \mathfrak{L} \otimes \mathfrak{L}$.

(2) All Lie super-bialgebra structures on \mathfrak{L} are triangular coboundary.

3 The proof of Theorem 2.3

The proof of Theorem 2.3 mainly depends on the following proposition.

Proposition 3.1. $\text{Der}(\mathfrak{L}, \mathfrak{V}) = \text{Inn}(\mathfrak{L}, \mathfrak{V})$.

Denote $\mathbb{Z}_s = \Gamma \cup (s + \Gamma)$, which is an abelian subgroup of \mathbb{R} . A Lie superalgebra L is called \mathbb{Z}_s -graded if $L = \bigoplus_{r \in \mathbb{Z}_s} L_r$ and $[L_p, L_q] \subset L_{p+q}$. Then the algebra \mathfrak{L} is \mathbb{Z}_s -graded with $\mathfrak{L}_p = \mathbb{C}L_p \oplus \mathbb{C}I_p \oplus \mathbb{C}G_p \oplus \mathbb{C}H_p$ if $s \in \Gamma$ and $\mathfrak{L}_p = \mathbb{C}L_p \oplus \mathbb{C}I_p$, $\mathfrak{L}_{p+s} = \mathbb{C}G_{p+s} \oplus \mathbb{C}H_{p+s}$ if $s \notin \Gamma$, $2s \in \Gamma$. Denote $\mathfrak{V}_r = \bigoplus_{p+q=r} \mathfrak{L}_p \otimes \mathfrak{L}_q$. Then \mathfrak{V} is a \mathbb{Z}_s -graded vector space. For any $r \in \mathbb{Z}_s$, denote

$$\text{Der}_r(\mathfrak{L}, \mathfrak{V}) = \{d \in \text{Der}(\mathfrak{L}, \mathfrak{V}) \mid d(\mathfrak{L}_p) \subset \mathfrak{V}_{p+r}, \quad \forall p \in \mathbb{Z}_s\}, \quad (3.1)$$

An element $d \in \text{Der}_r(\mathfrak{L}, \mathfrak{V})$ is called a *homogeneous derivation of degree r* , usually denoted by d_r . Similarly, we can define $\text{Inn}_r(\mathfrak{L}, \mathfrak{V})$, whose elements are called *homogeneous inner derivations of degree r* . Then $\text{Der}(\mathfrak{L}, \mathfrak{V}) = \prod_{r \in \mathbb{Z}_s} \text{Der}_r(\mathfrak{L}, \mathfrak{V})$. For any $d = \sum_{r \in \mathbb{Z}_s} d_r \in$

$\text{Der}(\mathfrak{L}, \mathfrak{V})$, the formal sum on the right hand side is not necessarily finite, while for any $x \in \mathfrak{L}$, $d(x) = \sum_{r \in \mathbb{Z}_s} d_r(x)$, in which there are finitely many nonzero summands.

A homogeneous element $L_p \in \mathfrak{L}$ (Resp. I_q, G_r, H_t) is called a homogeneous element of degree p_L (Resp. q_I, r_G, t_H), and denoted by $\deg(L_p)$ (Resp. $\deg(I_q), \deg(G_r), \deg(H_t)$). Define an order for homogeneous elements in \mathfrak{L} as follows:

$$\deg(L_p) > \deg(I_q) > \deg(G_r) > \deg(H_t) \quad (3.2)$$

and $\deg(A_p) > \deg(A_q) \iff p > q$, $\deg(B_r) > \deg(B_t) \iff r > t$, for $p, q \in \Gamma, r, t \in s + \Gamma$, $A \in \{L, I\}$ and $B \in \{G, H\}$. Then define *degree of homogeneous elements $A_p \otimes B_q$ in \mathfrak{V}* by:

$$\deg(A_p \otimes B_q) = (\deg(A_p), \deg(B_q)), \quad (3.3)$$

where $A, B \in \{L, I, G, H\}$, $p, q \in \mathbb{Z}_s$.

Denote by \mathbf{L} (Resp. $\mathbf{I}, \mathbf{G}, \mathbf{H}$) the \mathbb{C} -vector space spanned by

$$\{L_p \text{ (Resp. } I_p, G_p, H_p) \mid p \in \mathbb{Z}_s\}.$$

Any $u \in \mathfrak{V}$ can be written as the following formal sum of homogeneous summands:

$$u = \sum a_{p,q}^{A,B} A_p \otimes B_q, \quad (3.4)$$

where A_p, B_q are homogeneous elements in $\{\mathbf{L}, \mathbf{I}, \mathbf{G}, \mathbf{H}\}$ and $a_{p,q}^{A,B} \in \mathbb{C}$.

Definition 3.2. For any nonzero element $u \in \mathfrak{V}$, with a formal sum of homogeneous summands given above, we define the degree of u as follows: $\deg(u) = \max\{\deg(X \otimes Y) \mid X \otimes Y \text{ is a homogeneous summand of } u \text{ given in (3.4) whose coefficient is nonzero}\}$.

Fix a positive element $\epsilon \in \Gamma$, and denote by \mathfrak{A} the subalgebra of \mathfrak{L} spanned by $\{L_{\epsilon k} \mid k \in \mathbb{Z}\}$ as a vector space over \mathbb{C} , which is isomorphic to the centerless Virasoro algebra Vir : $\{L_k \mid [L_m, L_n] = (m-n)L_{m+n}\}$ with the isomorphism $\sigma(L_k) = \epsilon^{-1}L_{\epsilon k}$.

Proposition 3.3. *If $x \in \mathfrak{L}$ satisfies $[L_{\epsilon m}, x] = 0$ for infinitely many $m > 0$ or $m < 0$, then $x = 0$.*

Proof. We first consider the case that there are infinitely many $m > 0$ satisfying $[L_{\epsilon m}, x] = 0$. If $x \neq 0$, we can write x in the form of linear combinations of homogeneous elements in \mathfrak{L} . The highest degree summand must be a nonzero multiple of A_p in which $A \in \{L, I\}$, $p \in \Gamma$ or $A \in \{G, H\}$, $p \in s + \Gamma$. From the definition of brackets given in (1.1), we can find a suitable $N > 0$, for all $m > N$, $[L_{\epsilon m}, A_p] \neq 0$. Then the highest degree summand of $[L_{\epsilon m}, x]$ is a nonzero multiple of $A_{\epsilon m+p}$, Contradiction! As for the case there are infinitely many $m < 0$ satisfying $[L_{\epsilon m}, x] = 0$, we can consider the lowest degree summands. We similarly get a contradiction. Thus $x = 0$. \square

Proposition 3.4. *If $u \in \mathfrak{V}$ satisfies $L_{\epsilon m} \circ u = 0$ for infinitely many $m > 0$ or $m < 0$, then $u = 0$.*

Proof. If $u \neq 0$, we can give a formal sum of u as in (3.4). If $L_{\epsilon m} \circ u = 0$ for infinitely many $m > 0$, one can consider the highest degree summand of u , which must be a nonzero multiple of $X \otimes Y$, where X, Y are homogeneous elements in $\{\mathbf{L}, \mathbf{I}, \mathbf{G}, \mathbf{H}\}$. The highest degree summand of $L_{\epsilon m} \circ u$ must be a nonzero multiple of $[L_{\epsilon m}, X] \otimes Y$, which implies $[L_{\epsilon m}, X] = 0$ for infinitely many $m > 0$. According to Proposition 3.3, we get $X = 0$. Contradiction! As for the case $L_{\epsilon m} \circ u = 0$ for infinitely many $m < 0$, we consider the lowest degree summand. We similarly get a contradiction. Thus $u = 0$. \square

The following proposition follows from Proposition 3.4.

Proposition 3.5. *An element $r \in \text{Im}(1 \otimes 1 - \tau) \in \mathfrak{V}$ satisfies CYBE in (2.3) if and only if it satisfies MYBE in (2.4).*

We first prove Theorem 2.3 for the case $s \in \Gamma$. Then $\mathfrak{L} = \text{Span}_{\mathbb{C}}\{L_p, I_p, G_p, H_p \mid p \in \Gamma\}$ admits the following non-vanishing Lie brackets

$$\begin{aligned} [L_p, L_q] &= (p-q)L_{p+q}, & [L_p, I_q] &= (p-q)I_{p+q}, \\ [L_p, H_q] &= (\frac{p}{2}-q)H_{p+q}, & [G_p, G_q] &= I_{p+q}, \\ [L_p, G_q] &= (\frac{p}{2}-q)G_{p+q}, & [I_p, G_q] &= (p-2q)H_{p+q}. \end{aligned} \tag{3.5}$$

It is easy to see that $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{L_0\}$ is the Cartan Subalgebra (CSA) of \mathfrak{L} . And

$$\mathfrak{L}_p = \{x \in \mathfrak{L} \mid [L_0, x] = -px\}.$$

Denote $\Gamma^* = \{t \in \Gamma \mid t \neq 0\}$.

Lemma 3.6. $\text{Der}_t(\mathfrak{L}, \mathfrak{V}) = \text{Inn}_t(\mathfrak{L}, \mathfrak{V}), \forall t \in \Gamma^*.$

Proof. For any $d_t \in \text{Der}_t(\mathfrak{L}, \mathfrak{V})$ ($t \in \Gamma^*$), we can write $d_t = d_{t,\bar{0}} + d_{t,\bar{1}}$, where $d_{t,\bar{i}} \in \text{Der}_{\bar{i}}(\mathfrak{L}, \mathfrak{V})$ for $i = 0, 1$. Using $[L_0, L_p] = -pL_p$, one can deduce

$$L_0 \circ d_{t,\bar{0}}(L_p) - L_p \circ d_{t,\bar{0}}(L_0) = -pd_{t,\bar{0}}(L_p),$$

which implies

$$-(p+t)d_{t,\bar{0}}(L_p) - L_p \circ d_{t,\bar{0}}(L_0) = -pd_{t,\bar{0}}(L_p), \quad d_{t,\bar{0}}(L_p) = L_p \circ \left(-\frac{d_{t,\bar{0}}(L_0)}{t}\right).$$

Using $[L_0, I_p] = -pI_p$, $[L_0, G_p] = -pG_p$ and $[L_0, H_p] = -pH_p$, we obtain

$$\begin{aligned} d_{t,\bar{0}}(I_p) &= I_p \circ \left(-\frac{d_{t,\bar{0}}(L_0)}{t}\right), \\ d_{t,\bar{0}}(G_p) &= G_p \circ \left(-\frac{d_{t,\bar{0}}(L_0)}{t}\right), \\ d_{t,\bar{0}}(H_p) &= H_p \circ \left(-\frac{d_{t,\bar{0}}(L_0)}{t}\right). \end{aligned}$$

Hence $d_{t,\bar{0}} = \left(-\frac{d_{t,\bar{0}}(L_0)}{t}\right)_{\text{Inn}}$. Similarly, $d_{t,\bar{1}} = \left(-\frac{d_{t,\bar{1}}(L_0)}{t}\right)_{\text{Inn}}$. Thus $d_t = \left(-\frac{d_{t,\bar{0}}(L_0)}{t} - \frac{d_{t,\bar{1}}(L_0)}{t}\right)_{\text{Inn}}$, which implies $\text{Der}_t(\mathfrak{L}, \mathfrak{V}) = \text{Inn}_t(\mathfrak{L}, \mathfrak{V}), \forall t \in \Gamma^*$. \square

Lemma 3.7. $d_0(L_0) = 0$ for $d_0 \in \text{Der}_0(\mathfrak{L}, \mathfrak{V})$.

Proof. Using $[L_0, L_p] = -pL_p$, we obtain

$$L_0 \circ d_0(L_p) - L_p \circ d_0(L_0) = -pd_0(L_p),$$

which implies

$$L_p \circ d_0(L_0) = 0, \quad \forall p \in \Gamma.$$

From Proposition 3.4, we know $d_0(L_0) = 0$. \square

Lemma 3.8. Replace d_0 by $d_0 - u_{\text{Inn}}$, where $u \in \mathfrak{V}_0$, this replacement does not affect the results we already obtain in Lemma 3.7. With a suitable replacement, we can suppose $d_0(L_\epsilon) = 0$.

Proof. To prove such a replacement does not affect the results we already obtain in Lemma 3.7, it is enough to prove $L_0 \circ u = 0$ for all $u \in \mathfrak{V}_0$, which is obvious from the diagonal action defined in (2.1) and the Lie brackets defined in (3.5).

Write $d_0 = d_{0,\bar{0}} + d_{0,\bar{1}}$, where $d_{0,\bar{0}} \in \text{Der}_{\bar{0}}(\mathfrak{L}, \mathfrak{V})$, $d_{0,\bar{1}} \in \text{Der}_{\bar{1}}(\mathfrak{L}, \mathfrak{V})$. One can suppose

$$\begin{aligned} d_{0,\bar{0}}(L_\epsilon) &= \sum_a e_a L_{a+\epsilon} \otimes L_{-a} + \sum_b e_b L_{b+\epsilon} \otimes I_{-b} + \sum_c e_c I_{c+\epsilon} \otimes L_{-c} \\ &\quad + \sum_d e_d I_{d+\epsilon} \otimes I_{-d} + \sum_a f_a G_{a+\epsilon} \otimes G_{-a} + \sum_b f_b G_{b+\epsilon} \otimes H_{-b} \\ &\quad + \sum_c f_c H_{c+\epsilon} \otimes G_{-c} + \sum_d f_d H_{d+\epsilon} \otimes H_{-d}, \\ d_{0,\bar{1}}(L_{-\epsilon}) &= \sum_a e_a L_{a-\epsilon} \otimes G_{-a} + \sum_b e_b G_{b-\epsilon} \otimes L_{-b} + \sum_c e_c L_{c-\epsilon} \otimes H_{-c} \\ &\quad + \sum_d e_d H_{d-\epsilon} \otimes L_{-d} + \sum_a f_a I_{a-\epsilon} \otimes G_{-a} + \sum_b f_b G_{b-\epsilon} \otimes I_{-b} \\ &\quad + \sum_c f_c I_{c-\epsilon} \otimes H_{-c} + \sum_d f_d H_{d-\epsilon} \otimes I_{-d}, \end{aligned}$$

where $a, b, c, d \in \Gamma$, $e_a, e_b, e_c, e_d, f_a, f_b, f_c, f_d \in \mathbb{C}$. Since $\mathbf{A} \otimes \mathbf{B}$ is invariant under the diagonal action of elements in \mathbf{L} , in which $\mathbf{A}, \mathbf{B} \in \{\mathbf{L}, \mathbf{I}, \mathbf{G}, \mathbf{H}\}$, it is equal to prove our lemma as follows: By replacing d_0 by $d_0 - u_{\text{Inn}}$, where $u \in \mathfrak{V}_0 \cap (\mathbf{A} \otimes \mathbf{B})$, we can suppose $d_0(L_\epsilon) \cap (\mathbf{A} \otimes \mathbf{B}) = 0$, in which $\mathbf{A}, \mathbf{B} \in \{\mathbf{L}, \mathbf{I}, \mathbf{G}, \mathbf{H}\}$.

Here we only consider $d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L})$, and all others are similar. Suppose

$$d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L}) = \sum_{\beta_j} \sum_i a_i L_{\beta_j+i\epsilon} \otimes L_{-\beta_j-(i-1)\epsilon},$$

$$d_0(L_{-\epsilon}) \cap (\mathbf{L} \otimes \mathbf{L}) = \sum_{\beta_j} \sum_i b_i L_{\beta_j+i\epsilon} \otimes L_{-\beta_j-(i+1)\epsilon},$$

in which $i \in \mathbb{Z}$, $\beta_i \in \Gamma$, $\beta_i \not\equiv \beta_j \pmod{\mathbb{Z}\epsilon}$ for $i \neq j$. So we can assume only one β appeared in this formal sum, i.e.,

$$d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L}) = \sum_{i=1}^m a_i L_{\beta+i\epsilon} \otimes L_{-\beta-(i-1)\epsilon},$$

$$d_0(L_{-\epsilon}) \cap (\mathbf{L} \otimes \mathbf{L}) = \sum_{i=1}^m b_i L_{\beta+i\epsilon} \otimes L_{-\beta-(i+1)\epsilon},$$

in which $m \in \mathbb{Z}_+$, $a_i, b_i \in \mathbb{C}$, $(a_1, b_1) \neq (0, 0)$, $(a_m, b_m) \neq (0, 0)$. Using $[L_\epsilon, L_{-\epsilon}] = 2\epsilon L_0$ and Lemma 3.7, we obtain

$$L_{-\epsilon} \circ d_0(L_\epsilon) = L_\epsilon \circ d_0(L_{-\epsilon}),$$

which implies

$$L_{-\epsilon} \circ \left(\sum_i^m a_i L_{\beta+i\epsilon} \otimes L_{-\beta-(i-1)\epsilon} \right) = L_\epsilon \circ \left(\sum_i^m b_i L_{\beta+i\epsilon} \otimes L_{-\beta-(i+1)\epsilon} \right).$$

Comparing the same degree summands on both sides, we obtain

$$a_1 b_1 \neq 0, \quad a_m b_m \neq 0, \quad b_m [L_\epsilon, L_{\beta+m\epsilon}] = 0, \quad a_1 [L_{-\epsilon}, L_{\beta+\epsilon}] = 0.$$

Recall the definition *length* of $u = \sum_{i=1}^m e_i L_{\beta+i\epsilon} \otimes L_{-\beta-(i-1)\epsilon}$. We say u have length m if $e_1 e_m \neq 0$. Replacing d_0 by $d_0 - u_{\text{Inn}}$, in which $u = \frac{a_m}{\beta+(m+1)\epsilon} L_{\beta+m\epsilon} \otimes L_{-(\beta+m\epsilon)}$, we successfully reduce the length of $d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L})$ at least one. By induction on the length of $d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L})$, we arrive at $d_0(L_\epsilon) \cap (\mathbf{L} \otimes \mathbf{L}) = 0$. Thus, we can assume $d_0(L_\epsilon) = 0$. \square

From the proof of Lemma 3.8, we immediately get the following proposition.

Proposition 3.9. $d_0(L_{-\epsilon}) = 0$.

Lemma 3.10. $d_0(L_{k\epsilon}) = 0, \forall k \in \mathbb{Z}$.

Proof. Since $L_\epsilon, L_{2\epsilon}, L_{-\epsilon}, L_{-2\epsilon}$ generate \mathfrak{A} , we only need to prove $d_0(L_{2\epsilon}) = 0 = d_0(L_{-2\epsilon})$. Using $[L_{-\epsilon}, L_{2\epsilon}] = -3\epsilon L_\epsilon$, Lemma 3.8 and Proposition 3.9, we obtain $L_{-\epsilon} \circ d_0(L_{2\epsilon}) = 0$. From the brackets defined in (1.1), it is easy to verify that the homogeneous element $u \in \mathfrak{V}$ satisfying $L_{-\epsilon} \circ u = 0$ must be a linear combination of $A \otimes B$, $A, B \in \{L_{-\epsilon}, I_{-\epsilon}, G_{-\frac{\epsilon}{2}}, H_{-\frac{\epsilon}{2}}\}$ (If $\frac{\epsilon}{2} \notin \Gamma$, then we treat $G_{-\frac{\epsilon}{2}} = H_{-\frac{\epsilon}{2}} = 0$), but none of them can be a summand of $d_0(L_{2\epsilon})$, which implies $d_0(L_{2\epsilon}) = 0$. Similarly, we get $d_0(L_{-2\epsilon}) = 0$. Then $d_0(L_{k\epsilon}) = 0, \forall k \in \mathbb{Z}$. \square

Lemma 3.11. $d_0(A_p) = 0$ where $A \in \{L, I, G, H\}$ and $p \in \mathbb{Z}$.

Proof. Using $[L_{m\epsilon}, [L_{-m\epsilon}, G_p]] = -(\frac{m\epsilon}{2} + p)(\frac{3m\epsilon}{2} - p)G_n$ and Lemma 3.10, we obtain

$$L_{m\epsilon} \circ (L_{-m\epsilon} \circ (d_0(G_p))) = -(\frac{m\epsilon}{2} + p)(\frac{3m\epsilon}{2} - p)d_0(G_p). \quad (3.6)$$

For a formal sum of $d_0(G_p) \neq 0$ as in (3.4), one can suppose its highest degree summand is a nonzero multiple of $A_{\beta+p} \otimes B_{-\beta}$ where $A, B \in \{L, I, G, H\}$. We can choose a suitable $m > 0$ (such m can always be found since it is easy to get a contradiction from Proposition 3.4) such that the highest degree summand of $L_{m\epsilon} \circ (L_{-m\epsilon} \circ (d_0(G_p)))$ is a nonzero multiple of $A_{\beta+p+m\epsilon} \otimes B_{-(\beta+m)\epsilon}$. We immediately get a contradiction by comparing the highest degree summands on both sides of (3.6).

Similarly, $d_0(L_p) = d_0(I_p) = d_0(H_p) = 0$. Then this Lemma follows. \square

From Lemma 3.11 and the fact $\mathfrak{L} = \text{Span}_{\mathbb{C}}\{L_p, I_p, G_p, H_p | p \in \Gamma\}$ we have $\text{Der}_0(\mathfrak{L}, \mathfrak{V}) = \text{Inn}_0(\mathfrak{L}, \mathfrak{V})$. This, together with Lemma 3.6, gives the following proposition.

Proposition 3.12. For $s \in \Gamma$, $\text{Der}(\mathfrak{L}, \mathfrak{V}) = \text{Inn}(\mathfrak{L}, \mathfrak{V})$.

In the case $s \notin \Gamma$ and $2s \in \Gamma$, $\mathfrak{L} = \text{Span}_{\mathbb{C}}\{L_p, I_p, G_{p+s}, H_{p+s} | p \in \Gamma\}$ admits the following non-vanishing Lie brackets

$$\begin{aligned} [L_p, L_q] &= (p - q)L_{p+q}, & [L_p, I_q] &= (p - q)I_{p+q}, \\ [L_p, H_{q+s}] &= (\frac{p}{2} - q - s)H_{p+q}, & [G_{p+s}, G_{q+s}] &= I_{p+q+2s}, \\ [L_p, G_{q+s}] &= (\frac{p}{2} - q - s)G_{p+q+s}, & [I_p, G_{q+s}] &= (p - 2q - 2s)H_{p+q+s}. \end{aligned} \quad (3.7)$$

It is easy to see that $\mathfrak{h} := \text{Span}_{\mathbb{C}}\{L_0\}$ is the Cartan Subalgebra (CSA) of \mathfrak{L} and \mathfrak{L}_p can be given as follows

$$\mathfrak{L}_p = \{x \in \mathfrak{L} \mid [L_0, x] = -px, p \in \mathbb{Z}_s\}.$$

Using the similar arguments as those presented in the proof of Proposition 3.12, we can deduce the following results.

Lemma 3.13. $\text{Der}_t(\mathfrak{L}, \mathfrak{V}) = \text{Inn}_t(\mathfrak{L}, \mathfrak{V}), \forall t \in \mathbb{Z}_s^*.$

Lemma 3.14. For $d_0 \in \text{Der}_0(\mathfrak{L}, \mathfrak{V}), d_0(L_0) = 0.$

Lemma 3.15. Replace d_0 by $d_0 - u_{\text{Inn}}$, where $u \in \mathfrak{V}_0$, this replacement does not affect the results we already obtain in Lemma 3.14. With a suitable replacement, we can suppose $d_0(L_{\epsilon}) = 0.$

Lemma 3.16. $d_0(L_{-\epsilon}) = 0.$

Lemma 3.17. $d_0(L_{k\epsilon}) = 0, \forall k \in \mathbb{Z}.$

Lemma 3.18. $d_0(A_p) = 0$ where $A \in \{L, I\}, p \in \Gamma$ or $A \in \{G, H\}, p \in s + \Gamma.$

Proposition 3.19. For $s \notin \Gamma$ and $2s \in \Gamma, \text{Der}(\mathfrak{L}, \mathfrak{V}) = \text{Inn}(\mathfrak{L}, \mathfrak{V}).$

Proof of Theorem 2.3 Propositions 3.12 and 3.19 imply Proposition 3.1, which is a restatement of Theorem 2.3 (1). Theorem 2.3 (2) follows from Theorem 2.3 (1) and Proposition 3.5.

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